

# Conservation of Energy in Black Holes and in Cosmology

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## Abstract

We first review the various definition of the total energy in the gravitational system. The naive definition has some defects, and we review how to modify the definition of the total energy. Then we explicitly demonstrate how to calculate the total energy of the system. Our example is the total energy of a black hole in the expanding closed de Sitter universe in (2+1) dimension. In general, we find that the contribution to the total energy comes only from the singularity. Then we can calculate the total energy by evaluating the contribution around the singularity.

## 1 Introduction

It has a long history how to define the total energy in the gravitational theory. Because of the invariance under the general coordinate transformation, it becomes ambiguous how to define the time variable. Then the Hamiltonian, which is conjugate to time variable, becomes ambiguous. Especially there are many ways[1, 2, 3, 4] how to define the energy of the gravitational field. The purpose of this paper give the prescription to calculate the total energy of the system with local singularity. It is known that (i)  $E_{\text{total}} = (\text{finite})$  for the asymptotically flat space-time and (ii)  $E_{\text{total}} = 0$  for the closed universe.

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The contribution to the total energy comes only from the singularity, so we can calculate the total energy by evaluating the contribution around the singularity.

## 2 Energy of Gravitational Field - Review -

### 2.1 Action and the Einstein's Equation

The Hilbert-Einstein action is given by [5]

$$I = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_{\text{matter}}, \quad (1)$$

where  $\kappa = 8\pi G$ . Taking the variation with respect to  $g_{\mu\nu}$ , we have the Einstein's equation  $\delta I_{\text{HE}}/\delta g_{\mu\nu} = 0$ , which gives

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu}, \quad (2)$$

with the boundary condition

$$\int d^4x \partial_\mu \mathcal{B}^\mu = 0, \quad (3)$$

$$\mathcal{B}^\mu = \sqrt{-g} \left\{ g^{\mu\nu} (\delta \Gamma_{\nu\lambda}^\delta) - g^{\nu\lambda} (\delta \Gamma_{\nu\lambda}^\mu) \right\}. \quad (4)$$

Notice that the boundary term  $\mathcal{B}^\mu$  contains not only the variation of the metric  $\delta g_{\mu\nu}$  but also the variation of the first derivatives of the metric  $\delta g_{\mu\nu,\lambda}$  through the affine connection.

### 2.2 Einstein prescription

In the naive approach, the boundary term  $\mathcal{B}^\mu$  contains not only the variation of the metric but also the variation of the first derivatives of the metric, which makes difficult to find the classical solution compatible with boundary condition. There are various approaches to avoid this problem.

First we review the primitive expression of energy conservations derived by Einstein [1] in order to avoid the appearance of the variation of the derivative of the metric in the boundary. Using the identity

$$\sqrt{-g} R = \sqrt{-g} G + \partial_\mu \mathcal{D}^\mu, \quad (5)$$

$$G = g^{\mu\nu} \{ \Gamma_{\mu\nu}^\rho \Gamma_{\rho\lambda}^\lambda - \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\lambda}^\rho \}, \quad (6)$$

$$\mathcal{D}^\mu = \sqrt{-g} \{ g^{\mu\nu} \Gamma_{\nu\lambda}^\lambda - g^{\rho\sigma} \Gamma_{\rho\sigma}^\mu \}, \quad (7)$$

and the action is taken in the following form

$$I^* = \frac{1}{2\kappa} \int d^4x \sqrt{-g} G + \int d^4x \sqrt{-g} \mathcal{L}_{\text{matter}}. \quad (8)$$

After taking the variation of  $I^*$  with respect to  $g_{\mu\nu}$ , the same Einstein's equation Eq.(2) is obtained with new boundary condition

$$\int d^4x \partial_\mu \mathcal{B}^{*\mu} = 0, \quad (9)$$

$$\mathcal{B}^{*\mu} = -\sqrt{-g} \left\{ \delta(\sqrt{-g} g^{\mu\nu}) \Gamma_{\nu\lambda}^\delta - \delta(\sqrt{-g} g^{\nu\lambda}) \Gamma_{\nu\lambda}^\mu \right\}. \quad (10)$$

In this expression, the boundary term  $\mathcal{B}^{*\mu}$  contains only the variation of the metric as expected. Putting  $\mathcal{L}_G^* = \sqrt{-g} G / (2\kappa)$ , the energy-momentum tensor of the gravitational field is given by

$$u_\nu^\mu = \frac{\partial \mathcal{L}_G^*}{\partial g_{,\mu}^{\rho\sigma}} g_{,\mu}^{\rho\sigma} - \delta_\nu^\mu \mathcal{L}_G^*. \quad (11)$$

Then the conservation law of energy-momentum is expressed as

$$\partial_\mu (u_\nu^\mu + T_\nu^\mu) = 0, \quad (12)$$

where  $T_\nu^\mu$  denotes the energy-momentum tensor density of matter. However the separation of the dynamical metric from the auxiliary metric is not clear in this method.

### 2.3 ADM canonical formalism

In order to clarify the physical meaning of the metrics, we often use the ADM formalism [6] where we decompose the 4-dimensional space-time metric into (3+1) dimensional space and time metric. We consider the Cauchy surface  $\Sigma_t$  parametrized by a global function of time  $t$ . Let  $n^a$  be the unit normal vector to the hypersurface  $\Sigma_t$ . The metrics are decomposed into the form

$$ds^2 = -N^2 dt^2 + h_{ab} (dx^a + N^a) (dx^b + N^b), \quad (13)$$

$$g_{ab} = \begin{pmatrix} -N^2 + N^a N_b & N_a \\ N_a & h_{ab} \end{pmatrix}, \quad (14)$$

where  $N$  is the lapse function,  $N^a$  is the shift operator and  $h_{ab}$  is the spatial metric. In this expression, the determinant of the metrics is  $g = Nh$ .

First we proceed the Lagrangian formulation under the ADM (3+1) decomposition. Defining the extrinsic curvature of  $\Sigma_t$

$$K_{ab} = h_a^c \nabla_c n_b \quad (= N\Gamma_{ab}^0) \quad (15)$$

and its mean value  $K = K_a^a$ , the gravitational action is taken in the form

$$I_G = \frac{1}{2\kappa} \left( \int d^4x \sqrt{-g} R + \int d^3x \sqrt{h} 2K \right), \quad (16)$$

$$= \frac{1}{2\kappa} \int d^4x N \sqrt{h} (K_{ab} K^{ab} - K^2 + {}^{(3)}R). \quad (17)$$

The notation (3) in the left upper indices denotes the corresponding three dimensional quantities. Taking the variation of  $I_G$  we have the Einstein's equation with the boundary condition, which contains only the variation of the first time derivatives of metrics. The surface term  $\int d^4x \partial_t K / \kappa$  plays the role to make the action well defined in the Euclidean section [7].

Next we proceed to the Hamiltonian formulation. The canonical momentum and the Hamiltonian density are expressed as

$$\pi^{ab} = \frac{\partial \mathcal{L}_G}{\partial \dot{h}_{ab}} = \sqrt{h} (K^{ab} - K h^{ab}), \quad (18)$$

$$\begin{aligned} \mathcal{H}_G &= \pi^{ab} \dot{h}_{ab} - \mathcal{L}_G \\ &= NH + N^a P_a + 2D_a (h^{-1/2} N_b \pi^{ab}). \end{aligned} \quad (19)$$

In the above expression, the last term contribute only to an unimportant boundary term and will be dropped in the following. The Hamiltonian function and momentum function are given by

$$H = \frac{1}{\sqrt{h}} (\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2) - \sqrt{h} {}^{(3)}R, \quad (20)$$

$$P_a = -2\sqrt{h} (D_a h^{-1/2} \pi^{ab}). \quad (21)$$

Because the time derivatives of  $N$  and  $N^a$  do not appear in the Lagrangian, they are auxiliary variables. Then we must put the Hamiltonian and the momentum constraints as

$$H \approx 0, \quad (22)$$

$$P_a \approx 0. \quad (23)$$

The dynamical equations, which are derived by the canonical formalism, are given by

$$\dot{h}_{ab} = \frac{\delta H_G}{\delta \pi^{ab}} \equiv A_{ab}, \quad (24)$$

$$\dot{\pi}^{ab} = -\frac{\delta H_G}{\delta h_{ab}} \equiv -B_{ab}, \quad (25)$$

where

$$H_G = \int d^3x (NH + N^a P_a). \quad (26)$$

While, if we directly take the variation of this ADM Hamiltonian

$$\delta H_G = \int d^3x (A_{ab} \delta \pi^{ab} + B^{ab} \delta h_{ab}) - \delta C, \quad (27)$$

we obtain an extra surface contribution  $\delta C$ . So the dynamical equation derived by the canonical formalism and the dynamical equation derived by the Lagrangian formalism give the different boundary condition.

In order to avoid the additional surface term  $\delta C$ , Regge-Teitelboim [8] modify the total Hamiltonian in such a way as the dynamical equation, derived by the canonical formalism, is equivalent to the equation derived by the variation of the total Hamiltonian. The explicit form of  $\delta C$  is

$$\begin{aligned} \delta C = & \oint d^2 S_l G^{ijkl} (N \delta h_{ij,k} - N_{,k} \delta h_{ij}) \\ & + \oint d^2 S_l \{ 2N_k \delta \pi^{kl} + (2N^k \pi^{jl} - N^l \pi^{jk}) \delta h_{jk} \}, \end{aligned} \quad (28)$$

where

$$G^{ijkl} = \frac{1}{2} h^{1/2} (h^{ik} h^{jl} + h^{il} h^{jk} - 2h^{ij} h^{kl}). \quad (29)$$

If this surface term would vanish,  $H_G$  would be the correct Hamiltonian. However for the stationary asymptotically flat space-time ( $h_{ij} - \delta_{ij} \sim 1/r$ ), the first surface term is different from zero, that is

$$\oint d^2 S_l G^{ijkl} (N \delta h_{ij,k} - N_{,k} \delta h_{ij}) \sim \delta \oint d^2 S_l (h_{il,i} - h_{ii,l}). \quad (30)$$

A new gravitational Hamiltonian is defined by [9]

$$H'_G = H_G + E[h_{ij}], \quad (31)$$

where additional contribution comes from the surface term by

$$E[h_{ij}] = \oint d^2 S_l G^{ijkl} h_{ij,k}. \quad (32)$$

This is called ADM energy for the stationary asymptotically flat space-time. We get for the variation in this new gravitational Hamiltonian

$$\delta H'_G = \int d^3 x (A_{ab} \delta \pi^{ab} + B^{ab} \delta h_{ab}), \quad (33)$$

and the correct equations of motion Eq.(24)-Eq.(25) are recovered.

The value of the energy becomes in the following. First, one has  $H_G = 0$  because of the constraint equations Eq.(20)-Eq.(21). For the surface contribution one has  $E[h_{ij}] = 0$  for the closed universe, because there is no boundary, and  $E[h_{ij}] = (\text{finite})$  for the asymptotically flat space-time. For example, for the Schwarzschild metric,

$$ds^2 \sim -\left(1 - \frac{2Gm}{r}\right)dt^2 + (\delta_{ij} + 2Gm \frac{x^i x^j}{r^3})dx^i dx^j,$$

one has well known result  $E[h_{ij}] = m$  [4, 10].

## 2.4 Comment on the surface term

The surface terms in Eq.(16) and Eq.(32) are important for the Lagrangian formalism and the Hamiltonian formalism. Here we comment these surface terms are related to the surface term appeared in the identity Eq.(5)-Eq.(7). A. The surface term, which is the mean value of the scalar curvature in Eq.(16), plays a similar role to the time component of the surface term Eq.(7) in Einstein's prescription in order to avoid the time derivative contribution in the surface term;

$$h^{1/2} 2K \sim \mathcal{D}^t.$$

B. Using the three dimensional version of the identity Eq.(5) -Eq.(7)

$$\sqrt{h}^{(3)} R = \sqrt{h}^{(3)} G + \partial_l^{(3)} \mathcal{D}^l, \quad (34)$$

$$\begin{aligned} {}^{(3)}\mathcal{D}^l &\equiv \sqrt{h} \{ -h^{li} {}^{(3)}\Gamma_{ij}^j + h^{ij} {}^{(3)}\Gamma_{ij}^l \} \\ &= {}^{(3)}G^{ijkl} h_{ij,k}, \end{aligned} \quad (35)$$

where

$${}^{(3)}G^{ijkl} \equiv \frac{1}{2} h^{1/2} \{ h^{ik} h^{jl} + h^{il} h^{jk} - 2h^{ij} h^{kl} \}, \quad (36)$$

the additional energy Eq.(32) contributed from the surface term is simply expressed as

$$E[h_{ij}] = \frac{1}{2\kappa} \int d^3x \partial_l^{(3)} \mathcal{D}^l. \quad (37)$$

From the above observation the contribution to the total energy is interpreted in the following way. The term  $\sqrt{-g}R$  includes the singular terms coming from second derivative terms, and  $\sqrt{-g}G$  contains the less singular terms including only first derivative terms and the term  $\partial_\mu \mathcal{D}^\mu$  picks up the singular part as surface terms.

Next we study the cases of neither closed space nor asymptotically flat space-time.

### 3 Energy of System with Conical Singularity

#### 3.1 (2+1) dimensional conical singularity

We consider the system of the point particle with mass  $m_0$  [11]. Static spherically symmetric metric is given by

$$ds^2 = -dt^2 + e^{\sigma(\mathbf{r})} (dr^2 + r^2 d\phi^2). \quad (38)$$

Einstein equation becomes

$$\Delta\sigma(\mathbf{r}) = m_0 \delta(\mathbf{r}).$$

Then the solution is given as

$$ds^2 = -dt^2 + r^{-8Gm_0}(dr^2 + r^2 d\phi^2), \quad (39)$$

where the ranges of variables are  $0 \leq r < \infty$  and  $0 \leq \phi \leq 2\pi$ . After making the change of variables

$$\bar{r} = r^{1-4Gm_0}/(1-4Gm_0), \quad \bar{\phi} = (1-4Gm_0)\phi. \quad (40)$$

we get the flat metric

$$ds^2 = -dt^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2, \quad (41)$$

$$0 \leq \bar{r} < \infty, \quad 0 \leq \bar{\phi} \leq 2\pi - \Delta, \quad (42)$$

where the deficit angle  $\Delta = 8\pi Gm_0$  appears in the range of variable  $\phi$ . This metric is not asymptotically flat and the total energy formula Eq.(31) cannot apply. Here we define the total energy of this system as

$$E_{\text{total}} = -\frac{1}{2\kappa} \int d^3x H, \quad (43)$$

where  $H$  is defined in Eq.(20). Note that the minus sign, three dimensional integration and lack of the lapse function  $N$  in  $E_{\text{total}}$  of Eq.(43). Using the explicit expression for  $H$  in Eq.(20), the correct value of total energy is obtained [11]

$$E_{\text{total}} = \frac{1}{2\kappa} \int d^2x \sqrt{h}^{(2)} R = -\frac{1}{2\kappa} \oint dS_l \nabla^l \sigma(x) = m_0. \quad (44)$$

Next we explore the gravitational energy for the case with a singularity in the closed universe.

### 3.2 Closed and expanding de Sitter universe with conical singularity in (2+1) dimension

We consider the system of closed and expanding de Sitter universe with conical singularity. The classical solution is given by [12]

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{d\bar{r}^2}{1-\bar{r}^2} + (1-8Gm)\bar{r}^2 d\bar{\phi}^2 \right). \quad (45)$$



Here the scale factor of the universe  $a(t)$  is determined by the equation  $\dot{a}^2/a^2 = -1/a^2 + \lambda$ , and the solution is given by  $a(t) = \cosh(\sqrt{\lambda}t)/\sqrt{\lambda}$ . After making the change of variable  $\bar{\phi} = (1-b)\phi$ , the metrics in Eq.(45) shows the appearance of the deficit angle  $\Delta = 2\pi b \sim 8\pi Gm$ , where the parameter  $b = 1 - \sqrt{1 - 8Gm}$  is introduced. In order to evaluate the total energy of the system we make the change of variables

$$\bar{r} = r^{1-b}/(1-b),$$

and we get the metric in the form

$$ds^2 = -dt^2 + a(t)^2 r^{-2b} \left( \frac{dr^2}{1 - r^{2-2b}/(1-b)^2} + r^2 d\phi^2 \right). \quad (46)$$

The total gravitational energy is defined as the same form in Eq.(43). Almost all region in the integration vanish due to the constraint Eq.(22) and only contribution comes from the singular part around  $r = 0$ . Therefore the integration is performed small region  $\epsilon$  around  $r = 0$  and the total gravitational energy is obtained as

$$\begin{aligned} E_{\text{total}} &= -\frac{1}{2\kappa} \int \frac{1}{\sqrt{h}} (\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2) - \sqrt{h}^{(2)} ({}^{(2)}R - 2\lambda) \\ &= -\frac{1}{\kappa} \int_0^\epsilon dr \int_0^{2\pi} d\phi \frac{\partial}{\partial r} \left( \sqrt{h} h^{rr} \frac{\partial}{\partial r} \sqrt{h_{\phi\phi}} \right) \\ &= 2m, \end{aligned} \quad (47)$$

where  $\lambda$  is the cosmological constant. Note that the value of the total energy for the closed expanding de Sitter solution with conical singularity in Eq.(47) is twice for the simple conical singular solution in Eq.(44).

The gravitational energy in asymptotically de Sitter spaces is examined by Abbott and Deser [13]. The three dimensional cosmological gravity is studied by Deser and Jackiw and found static many-body solutions [14]. Mass in static asymptotically de Sitter spaces was studied by V. Balasubramanian et al. [15].

## 4 Summary and Discussion

We first review the derivation of the total gravitational energy, i) Einstein prescription, ii) ADM canonical formalism using Regge-Teitelboim deriva-

tion.

In section 3, we explicitly calculate the total gravitational energy for neither closed space nor stationary asymptotically flat space-time. Examples are the following (2+1) dimensional systems, i) system with conical singularity, ii) closed and expanding de Sitter universe with conical singularity.

In general, we find that the contribution to the total energy comes only from the singularity. Then we can calculate the total energy by evaluating the contribution around the singularity. We will show the complete proof for our statement near future.

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